3

HANDBOOK OF MATHEMATICAL FUNCTIONS

WITH FORMULAS, GRAPHS, AND MATHEMATICAL TABLES

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4466

The text relating to physical constants and conversion factors (page 6) has been modified to take into account the newly adopted Système Internationale d'Unites (SI).

ERRATA NOTICE

The original printing of this Handbook (June 1964) contained errors that have been corrected in the reprinted editions. These corrections are marked with an asterisk (*) for identification. The errors occurred on the following pages: 2–3, 6–8, 10, 15, 19–20, 25, 76, 85, 91, 102, 187, 189–197, 218, 223, 225, 233, 250, 255, 260–263, 268, 271–273, 292, 302, 328, 332, 333–337, 362, 365, 415, 423, 438–440, 443, 445, 447, 449, 451, 484, 498, 505–506, 509–510, 543, 556, 558, 562, 571, 595, 599, 600, 722–723, 739, 742, 744, 746, 752, 756, 760–765, 774, 777–785, 790, 797, 801, 822–823, 832, 835, 844, 886–889, 897, 914, 915, 920, 930–931, 936, 940–941, 944–950, 953, 960, 963, 989–990, 1010, 1026.

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26.5. Incomplete Beta Function

26.5.1

$$I_x(a,b)=\frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt \qquad (0\leq x\leq 1)$$

26.5.2

$$I_x(a,b)=1-I_{1-x}(b,a)$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions **26.4.1** with ν_1 and ν_2 degrees of freedom respectively, then $\frac{X_1^2}{X_1^2+X_2^2}$ is said to follow a beta distribution with ν_1 and ν_2 degrees of freedom and has the distribution function

26.5.3

$$P\left\{\frac{X_1^2}{X_1^2+X_2^2}\leq x\right\}=\frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt$$
$$=I_x(a,b) \qquad a=\frac{\nu_1}{2},\ b=\frac{\nu_2}{2}$$

Series Expansions ($0<x<1$)

26.5.4

$$* \quad I_x(a,b)=\frac{x^a(1-x)^b}{aB(a,b)}\left\{1+\sum_{n=0}^{\infty}\frac{B(a+1,n+1)}{B(a+b,n+1)}x^{n+1}\right\}$$

26.5.5

$$I_x(a,b)=\frac{x^a(1-x)^{b-1}}{aB(a,b)}\left\{1+\sum_{n=0}^{\infty}\frac{B(a+1,n+1)}{B(b-n-1,n+1)}\left(\frac{x}{1-x}\right)^{n+1}\right\}$$
$$=\frac{x^a(1-x)^{b-1}}{aB(a,b)}\left\{1+\sum_{n=0}^{s-2}\frac{B(a+1,n+1)}{B(b-n-1,n+1)}\left(\frac{x}{1-x}\right)^{n+1}\right\}+I_x(a+s,b-s)$$

26.5.6

$$1-I_x(a,b)=I_{1-x}(b,a)$$
$$=\frac{(1-x)^b}{B(a,b)}\sum_{i=0}^{a-1}(-1)^i\binom{a-1}{i}\frac{(1-x)^i}{b+i} \quad \text{(integer } a\text{)}$$

26.5.7

$$1-I_x(a,b)=I_{1-x}(b,a)$$
$$=(1-x)^{a+b-1}\sum_{i=0}^{a-1}\binom{a+b-1}{i}\left(\frac{x}{1-x}\right)^i \quad \text{(integer } a\text{)}$$

Continued Fractions

26.5.8

$$I_x(a,b)=\frac{x^a(1-x)^b}{aB(a,b)}\left\{\frac{1}{1+}\,\frac{d_1}{1+}\,\frac{d_2}{1+}\cdots\right\} \quad *$$

$$d_{2m+1}=-\frac{(a+m)(a+b+m)}{(a+2m)(a+2m+1)}x$$

$$d_{2m}=\frac{m(b-m)}{(a+2m-1)(a+2m)}x$$

Best results are obtained when $x<\frac{a-1}{a+b-2}$. Also the $4m$ and $4m+1$ convergents are less than $I_x(a,b)$ and the $4m+2$, $4m+3$ convergents are greater than $I_x(a,b)$.

26.5.9

$$I_x(a,b)=\frac{x^a(1-x)^{b-1}}{aB(a,b)}\left[\frac{e_1}{1+}\,\frac{e_2}{1+}\,\frac{e_3}{1+}\cdots\right]$$

$$* \quad x<1 \qquad e_1=1$$

$$e_{2m}=-\frac{(a+m-1)(b-m)}{(a+2m-2)(a+2m-1)}\,\frac{x}{1-x}$$

$$e_{2m+1}=\frac{m(a+b-1+m)}{(a+2m-1)(a+2m)}\,\frac{x}{1-x}$$

Recurrence Relations

26.5.10

$$I_x(a,b)=xI_x(a-1,b)+(1-x)I_x(a,b-1)$$

26.5.11

$$I_x(a,b)=\frac{1}{x}\{I_x(a+1,b)-(1-x)I_x(a+1,b-1)\}$$

26.5.12

$$\left[I_x(a,b)=\right]\frac{1}{a(1-x)+b}\{bI_x(a,b+1)+a(1-x)I_x(a+1,b-1)\} \quad *$$

26.5.13

$$I_x(a,b)=\frac{1}{a+b}\{aI_x(a+1,b)+bI_x(a,b+1)\}$$

26.5.14

$$I_x(a,a)=\frac{1}{2}I_{1-x'}\left(a,\frac{1}{2}\right), \qquad x'=4\left(x-\frac{1}{2}\right)^2\left[, x\leq\frac{1}{2}\right]^*$$

26.5.15

$$I_x(a,b)=\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^{b-1}+I_x(a+1,b-1)$$

26.5.16

$$I_x(a,b)=\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^b+I_x(a+1,b)$$

*See page II.

Asymptotic Expansions

26.5.17

$$1-I_x(a,b)=I_{1-x}(b,a)\sim\frac{\Gamma(b,y)}{\Gamma(b)}$$

$$-\frac{1}{24N^2}\left\{\frac{y^be^{-y}}{(b-2)!}(b+1+y)\right\}$$

$$+\frac{1}{5760N^4}\left\{\frac{y^be^{-y}}{(b-2)!}[(b-3)(b-2)(5b+7)(b+1+y)-(5b-7)(b+3+y)y^2]\right\}$$

$$y=-N\ln x,\qquad N=a+\frac{b}{2}-\frac{1}{2}$$

26.5.18

$$I_x(a,b)\sim\frac{\Gamma(a,w)}{\Gamma(a)}+\frac{e^{-w}w^a}{\Gamma(a)}\left\{\frac{(a-1-w)}{2b}\right.$$

$$\left.+\frac{1}{(2b)^2}\left(\frac{a^3}{2}-\frac{5}{3}a^2+\frac{3}{2}a-\frac{1}{3}-w\left[\frac{3}{2}a^2-\frac{11}{6}a+\frac{1}{3}\right]+w^2\left(\frac{3}{2}a-\frac{1}{6}\right)-\frac{1}{2}w^3\right)\right\}$$

$$w=b\left(\frac{x}{1-x}\right)$$

26.5.19

$$I_x(a,b)\sim P(y)-Z(y)\left[a_1+\frac{a_2(y-a_1)}{1+a_2}+\frac{a_3(1+y^2/2)}{1+a_2}+\ldots\right]$$

$$a_1=\frac{2}{3}(b-a)[(a+b-2)(a-1)(b-1)]^{-1/2}$$

$$a_2=\frac{1}{12}\left[\frac{1}{a-1}+\frac{1}{b-1}-\frac{13}{a+b-1}\right]$$

$$a_3=-\frac{8}{15}\left[a_1\left(a_2+\frac{3}{a+b-2}\right)\right]$$

$$y^2=2\left[(a+b-1)\ln\frac{a+b-1}{a+b-2}+(a-1)\ln\frac{a-1}{(a+b-1)x}+(b-1)\ln\frac{b-1}{(a+b-1)(1-x)}\right]$$

and y is taken negative when $x<\frac{a-1}{a+b-2}$

Approximations

26.5.20 If $(a+b-1)(1-x)\leq .8$

$$I_x(a,\,b)=Q(\chi^2|\nu)+\epsilon,$$

$$|\epsilon|<5\times10^{-3}\text{ if }a+b>6$$

$$\chi^2=(a+b-1)(1-x)(3-x)-(1-x)(b-1),$$

$$\nu=2b$$

26.5.21 If $(a+b-1)(1-x)\geq .8$

$$I_x(a,\,b)=P(y)+\epsilon,$$

$$|\epsilon|<5\times10^{-3}\text{ if }a+b>6$$

$$y=\frac{3\left[w_1\left(1-\frac{1}{9b}\right)-w_2\left(1-\frac{1}{9a}\right)\right]}{\left[\frac{w_1^2}{b}+\frac{w_2^2}{a}\right]^{\frac{1}{2}}},$$

$$w_1=(bx)^{1/3},\ w_2=[a(1-x)]^{1/3}$$

Approximation to the Inverse Function

26.5.22 If $I_{x_p}(a,\,b)=p$ and $Q(y_p)=p$ then

$$x_p\approx\frac{a}{a+be^{2w}}$$

$$w=\frac{y_p(h+\lambda)^{\frac{1}{2}}}{h}-\left(\frac{1}{2b-1}-\frac{1}{2a-1}\right)\left(\lambda+\frac{5}{6}-\frac{2}{3h}\right)$$

$$h=2\left(\frac{1}{2a-1}+\frac{1}{2b-1}\right)^{-1},\qquad\lambda=\frac{y_p^2-3}{6}$$

Relations to Other Functions and Distributions

	Function	*Relation*
26.5.23	Hypergeometric function	$\frac{1}{B(a,\,b)}\frac{x^a}{a}F(a,\,1-b;\,a+1;\,x)=I_x(a,\,b)$
26.5.24	Binomial distribution	$\sum_{s=a}^{n}\binom{n}{s}p^s(1-p)^{n-s}=I_p(a,n-a+1)$
26.5.25	"	$\binom{n}{a}p^a(1-p)^{n-a}=I_p(a,\,n-a+1)-I_p(a+1,\,n-a)$ *
26.5.26	Negative binomial distribution	$\sum_{s=a}^{n}\binom{n+s-1}{s}p^nq^s=I_q(a,\,n)$
26.5.27	Student's distribution	$\frac{1}{2}[1-A(t\|\nu)]=\frac{1}{2}I_x\left(\frac{\nu}{2},\frac{1}{2}\right),\qquad x=\frac{\nu}{\nu+t^2}$ *
26.5.28	F-(variance-ratio) distribution	$Q(F\|\nu_1,\nu_2)=I_x\left(\frac{\nu_2}{2},\frac{\nu_1}{2}\right),\qquad x=\frac{\nu_2}{\nu_2+\nu_1F}$

*See page II.